

LEGENDRIAN KNOTS IN LENS SPACES

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ABSTRACT. In this note, we first classify all topological torus knots lying on the Heegaard torus in Lens spaces, and then we classify Legendrian representatives of torus knots. We show that all Legendrian torus knots in universally tight contact structures on Lens spaces are determined up to contactomorphism by their knot type, rational Thurston-Bennequin invariant and rational rotation number.

1. INTRODUCTION

A *Legendrian knot* in a contact 3-manifold is a knot which is everywhere tangent to the contact planes. Legendrian knots are very natural objects in contact 3-manifolds and they play an important role in the theory. Legendrian knots are used to distinguish contact structures [16], to detect topological properties of knots [19] and to detect overtwistedness of contact structures [9].

There have been some recent progress in the classification of Legendrian knots in tight contact structures after the classification of Legendrian unknots done by Eliashberg and Fraser [5] and the classification of Legendrian torus knots and the figure eight knot done by Etnyre and Honda [8]. Legendrian knots in a cabled knot type are studied in [10] and complete classification is given in [20]. Recently, Legendrian twists knots are classified in [11]. Legendrian knots in 3-manifolds other than S^3 are also studied. For example, in [13], Legendrian linear curves on 3-torus T^3 are classified and in [1], Legendrian rational unknots in Lens spaces are classified.

In this note, we study Legendrian knots in Lens spaces. We focus on a class of knots called torus knots. *Torus knots* are knots that lie on the Heegaard torus without any points of intersection. First of all, not all torus knots in Lens spaces are null-homologous but all are rationally null-homologous. In Section 2, we study topological properties of torus knots. First, we find constrain on when a torus knot is null-homologous. Next, we compute the group of torus knots. By studying the diffeotopy group of Lens spaces, we completely classify all torus knots up to isotopy. Lastly, we construct a rational Seifert surface for a torus knot and we calculate its Euler characteristic. In section 3, we give a review of basic concepts in convex surface theory and we fix notation. In Section 4, by using convex surface theory tools, we

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study contact geometric properties of Legendrian torus knots. We study Legendrian representatives of torus knots in universally tight contact structures on Lens spaces. With the help of the rational Seifert surfaces that we constructed for torus knots, we calculate the rational Thurston-Bennequin invariants and the rational rotation numbers of Legendrian torus knots. By following the strategy outlined in [8], we first classify Legendrian torus knots with maximal rational Thurston-Bennequin invariant, and then we show that all Legendrian torus knots without maximal rational Thurston-Bennequin invariant destabilize and finally we determine the relationship between their stabilizations. We prove:

Theorem 1.1. *Legendrian torus knots in universally tight contact structures on a Lens space $L(p, q)$ are determined up to contactomorphism by their knot type, rational Thurston-Bennequin invariant and rational rotation number.*

In last section, we list several remarks and questions related to Legendrian knots in Lens spaces. We conclude:

Theorem 1.2. *Transverse torus knots in universally tight contact structures on Lens spaces are determined up to contactomorphism by their knot type and rational self-linking number.*

2. TOPOLOGICAL TORUS KNOTS IN LENS SPACES

For fixed relatively prime integers $p > q > 0$, let (V_1, V_2) be the genus 1 Heegaard splitting of a Lens space $L(p, q)$ which is described as

$$L(p, q) = V_1 \cup_{\varphi} V_2$$

where V_1 and V_2 are both $D^2 \times S^1$. The gluing map $\varphi : \partial V_1 \rightarrow \partial V_2$ is an orientation reversing map given by the matrix

$$\begin{pmatrix} -q & q' \\ p & p' \end{pmatrix}$$

with $pq' + qp' = 1$. In particular, the image of the meridian μ_1 of ∂V_1 is the curve $-q\mu_2 + p\lambda_2$ in ∂V_2 .

Let $K_{(a,b)}$ curve on the Heegaard torus ∂V_2 be a curve wraps a times in the meridional direction and b times in the longitudinal direction on ∂V_2 . If a and b are relatively prime then $K_{(a,b)}$ curve is a knot, in this case $K_{(a,b)}$ is called a (a, b) -torus knot in the Lens space $L(p, q)$.

Let μ_2 and λ_2 be a positive meridian-longitude basis for $H_1(\partial V_2)$. Then the torus knot $K_{(a,b)}$ can be written as $a[\mu_2] + b[\lambda_2]$ in homology. Note that any knot in a Lens space $L(p, q)$ is rationally null-homologous. Let r be the order of $K_{(a,b)}$ in $L(p, q)$, that is r is the smallest integer such that $r[K] = 0$ in $H_1(L(p, q)) = \mathbb{Z}_p$. For a knot $K_{(a,b)}$ of order r in $L(p, q)$, $p \mid rb$. Note also that not all torus knots in $L(p, q)$ are null-homologous. The torus knot $K_{(a,b)}$ is null-homologous if and only if $p \mid b$. Furthermore, considering the

corresponding meridional curves of the Heegaard splitting on ∂V_2 , for any torus knot $K_{(a,b)}$ of order r we have

$$r[K] = m[\mu_1] + l[\mu_2] = m(-q[\mu_2] + p[\lambda_2]) + l[\mu_2] \in H_1(\partial V_2).$$

where $m = \frac{rb}{p}$, $l = ra + mq = ra + \frac{rb}{p}q$. In the case when $K_{(a,b)}$ is null-homologous, we have $r = 1$.

Proposition 2.1. *Let $K_{(a,b)}$ be a (a,b) -torus knot on the Heegaard torus ∂V_2 in lens space $L(p,q)$.*

(1) *The group of a torus knot $K_{(a,b)}$ can be presented as*

$$\pi_1(L(p,q) - K_{(a,b)}) = \langle u, v \mid u^b = v^{pa+qb} \rangle$$

(2) *Two torus knots $K_{(a,b)}$ and $K_{(a',b')}$ have isomorphic groups if and only if $|b| = |b'|$ and $|pa + qb| = |pa' + qb'|$ or $|b| = |pa' + qb'|$ and $|b'| = |pa + qb|$.*

Proof. The complement of a neighborhood $\nu(K_{(a,b)})$ of a torus knot $K_{(a,b)}$ in $L(p,q)$ is the union of two solid tori glued along an annulus A where the core C of the annulus A is isotopic to the torus knot $K_{(a,b)}$. Namely, $L(p,q) \setminus K_{(a,b)} = \tilde{V}_1 \cup \tilde{V}_2$ where $\tilde{V}_i = \overline{V_i \setminus \nu(K_{(a,b)})}$ two solid tori, $i = 1, 2$, glued along the annulus $A = (L(p,q) \setminus K_{(a,b)}) \cap \partial V_2$.

Let $\tilde{\mu}_i$ and $\tilde{\lambda}_i$ be meridian and longitude pair for \tilde{V}_i where $\tilde{\mu}_i$ and $\tilde{\lambda}_i$ represent the trivial element and a generator of $\pi_1(\tilde{V}_i)$, respectively. For convenience we use the multiplicative notation for the fundamental group. We want to remark that throughout the paper we use additive notation for the homology group. Note that $[C] = [\tilde{\mu}_1]^{-p'a+q'b}[\tilde{\lambda}_1]^{pa+qb} = [\tilde{\lambda}_1]^{pa+qb}$ since $K_{(a,b)}$ is on ∂V_2 and

$$\begin{pmatrix} -q & q' \\ p & p' \end{pmatrix}^{-1} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -p' & q' \\ p & q \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -p'a + q'b \\ pa + qb \end{pmatrix}$$

Also, $[C] = [\tilde{\mu}_2]^a[\tilde{\lambda}_2]^b = [\tilde{\lambda}_2]^b$. Then, by Seifert-van Kampen theorem,

$$\pi_1(K_{(a,b)}) = \langle u, v \mid u^b = v^{pa+qb} \rangle$$

where $u = [\tilde{\lambda}_1]$ and $v = [\tilde{\lambda}_2]$. This proves (1).

Note that the subgroup $\langle u^b \rangle$ generates the centre of the knot group $\pi_1(K_{(a,b)})$ and $\pi_1(K_{(a,b)}) / \langle u^b \rangle = \mathbb{Z}_{|b|} * \mathbb{Z}_{|pa+qb|}$. Note also that u and v generate non-conjugate maximal finite cyclic subgroups of order $|b|$ and $|pa + qb|$ of $\mathbb{Z}_{|b|} * \mathbb{Z}_{|pa+qb|}$, respectively. Therefore, if $K_{(a,b)}$ and $K_{(a',b')}$ have isomorphic groups, then $|b| = |b'|$ and $|pa + qb| = |pa' + qb'|$ or $|b| = |pa' + qb'|$ and $|b'| = |pa + qb|$, proving (2). \square

Lemma 2.2. (1) *If $K_{(a,b)}$ and $K_{(a',b')}$ are two null-homologous torus knots in $L(p,q)$ that have isomorphic groups, then (a',b') is equal to one of the following pairs:*

- $(a, b), (-a, -b), B = (\frac{-2qb-pa}{p}, b), -B = (\frac{2qb+pa}{p}, -b),$
 $C = (\frac{b-qp a-q^2 b}{p}, pa+qb), -C = (\frac{-b+qp a+q^2 b}{p}, -pa-qb),$
 $D = (\frac{b+qp a+q^2 b}{p}, -pa-qb), -D = (\frac{-b-qp a-q^2 b}{p}, pa+qb),$
- (2) If $K_{(a,b)}$ and $K_{(a',b')}$ are two rationally null-homologous but not null-homologous torus knots in $L(p, q)$ that have isomorphic groups, then (a', b') is equal to one of the following pairs in the following cases:
 - $(a, b), (-a, -b)$ if $p \neq 2$ and $q^2 \not\equiv \pm 1 \pmod{p}$,
 - $(a, b), (-a, -b), C = (\frac{b-qp a-q^2 b}{p}, pa+qb), -C = (\frac{-b+qp a+q^2 b}{p}, -pa-qb)$ if $p \neq 2$ and $q^2 \equiv 1 \pmod{p}$,
 - $(a, b), (-a, -b), D = (\frac{b+qp a+q^2 b}{p}, -pa-qb), -D = (\frac{-b-qp a-q^2 b}{p}, pa+qb)$ if $p \neq 2$ and $q^2 \equiv -1 \pmod{p}$,
 - $(a, b), (-a, -b), (-b-a, b), (b+a, -b), (-a, 2a+b), (a, -2a-b), (a+b, -2a-b), (-a-b, 2a+b)$ if $p = 2$.

Proof. By Proposition 2.1(2) we know that $K_{(a,b)}$ and $K_{(a',b')}$ have isomorphic groups if and only if $|b| = |b'|$ and $|pa+qb| = |pa'+qb'|$ or $|b| = |pa'+qb'|$ and $|b'| = |pa+qb|$. Part (1) follows from the analysis of these cases. For part (2), we know that if $K_{(a,b)}$ is not null-homologous then $p \nmid b$. Therefore, when $p \neq 2$ the cases $(\frac{-2qb-pa}{p}, b), (\frac{2qb+pa}{p}, -b)$ do not occur and the cases $(\frac{b-qp a-q^2 b}{p}, pa+qb), (\frac{-b+qp a+q^2 b}{p}, -pa-qb)$ occur only if $p \mid (1-q^2)$. Similarly, the cases $(\frac{b+qp a+q^2 b}{p}, -pa-qb), (\frac{-b-qp a-q^2 b}{p}, pa+qb)$ occur only if $p \mid (1+q^2)$. The case when $p = 2$ and hence $q = 1$ is clear. \square

Let us now classify all topological torus knots on a Heegaard torus in Lens spaces up to isotopy. Recall that two knots K_1 and K_2 in a 3-manifold M are isotopic if there is a diffeomorphism $\phi : M \rightarrow M$ such that $\phi(K_1) = K_2$ and ϕ is isotopic to the identity map.

Theorem 2.3. *The torus knot $K_{(a,b)}$ is isotopic to $K_{(a',b')}$ in $L(p, q)$ if and only if $K_{(a',b')}$ is equal to one of the following pairs in the given cases:*

- (1) $\{K_{(a,b)}\}$ if $q \neq 1$ or $p = 1$,
- (2) $\{K_{(a,b)}, K_C = K_{(\frac{b-qp a-q^2 b}{p}, pa+qb)}\}$ if $p \neq 2$ and $q = 1$ or $p = 1$,
- (3) $\{K_{(a,b)}, K_{(-a,-b)}, K_C = K_{(-a, 2a+b)}, K_{-C} = K_{(a, -2a-b)}\}$ if $p = 2$.

For the proof of Theorem 2.3 we need the following theorem:

Theorem 2.4 (Bonahon [4, Theorem 3]). *The group of isotopy classes of diffeomorphisms of $L(p, q)$ for $p \geq 2$ is given by*

- (1) \mathbb{Z}_2 with generator τ if $q^2 \not\equiv \pm 1 \pmod{p}$
- (2) $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ with generator τ and σ_+ if $q^2 \equiv 1$ and $q \not\equiv \pm 1 \pmod{p}$
- (3) \mathbb{Z}_2 with generator τ if $q \equiv \pm 1 \pmod{p}$ and $p \neq 2$
- (4) \mathbb{Z}_4 with generator σ_- if $q^2 \equiv -1 \pmod{p}$ and $p \neq 2$
- (5) \mathbb{Z}_2 with generator σ_- if $p = 2$

Let (V_1, V_2) be the genus 1 Heegaard splitting of the Lens Space $L(p, q)$ defined as above. In Theorem 2.4, τ is the diffeomorphism that preserves each of the solid tori $V_i = D^2 \times S^1$ and acts by a complex conjugation on each factor of each Heegaard torus. Note that τ always exists and if $p = 2$, then τ is isotopic to the identity. In general, $L(p, q)$ does not admit a diffeomorphism that exchanges V_1 and V_2 except when $q^2 \equiv \pm 1 \pmod{p}$. If $q^2 \equiv 1 \pmod{p}$, there exists a diffeomorphism σ_+ that exchanges the Heegaard tori, namely $\sigma_+ : (u, v) \in V_1 \leftrightarrow (u, v) \in V_2$. If $q = 1$ or $p - 1$ then σ_+ is isotopic to the identity. Similarly, when $q^2 \equiv -1$, $L(p, q)$ admits a diffeomorphism σ_- that exchanges V_1 and V_2 and acts by complex conjugation on each V_i as follows: $\sigma_- : (u, v) \in V_1 \mapsto (\bar{u}, v) \in V_2$ and $(u, v) \in V_2 \mapsto (u, \bar{v}) \in V_1$. For diffeotopy groups of Lens spaces also see [14].

Proof. of Theorem 2.3. Let us first consider null-homologous knots case, not null-homologous case follows from the same argument. Let $K_{(a,b)}$ and $K_{(a',b')}$ be two isotopic null-homologous knots on the Heegaard torus ∂V_2 in $L(p, q)$. Since $K_{(a,b)}$ and $K_{(a',b')}$ have isomorphic groups, from Lemma 2.2, we know that the candidates for (a', b') are $(-a, -b)$, B , $-B$, C , $-C$, D and $-D$. Moreover, since $[K_{(a,b)}] = [K_{(a',b')}]$ in homology, the cases C , $-C$, D , $-D$ occur only if $q = 1$ or $p - 1$.

We first identify the diffeomorphisms that send $K_{(a,b)}$ to possible $K_{(a',b')}$'s and then we analyze when such diffeomorphisms are isotopic to the identity. Clearly, τ sends $K_{(a,b)}$ to $K_{(-a,-b)}$. Note that σ_+ sends $K_{(a,b)}$ on ∂V_2 to $K_{(a,b)}$ on ∂V_1 . Then after applying the gluing map $\phi : \partial V_1 \rightarrow \partial V_2$ with $pq' + qp' = 1$, we get

$$\begin{pmatrix} -q & q' \\ p & p' \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -qa + q'b \\ pa + p'b \end{pmatrix} = \begin{pmatrix} a' \\ b' \end{pmatrix}$$

Note that for $a' = -qa + q'b$ and $b' = pa + p'b$ we have $pa' + qb' = p(-qa + q'b) + q(pa + p'b) = (pq' + qp')b = b$. By Proposition 2.1(2), it follows that we are in the case when $b = pa' + qb'$ and $|b'| = |pa + qb|$. More precisely, $b = pa' + qb'$ and $b' = pa + p'b = pa + qb$ or $b' = pa + p'b = -pa - qb$. Since $(a, b) = 1$, the latter case does not occur. If we choose p' such that $qp' \equiv 1 \pmod{p}$, then we are left with the only case $b = pa' + qb'$ and $b' = pa + qb$ and in this case $(a', b') = (\frac{b-qp'a-q^2b}{p}, pa + qb) = C$. Therefore, σ_+ sends $K_{(a,b)}$ to K_C .

By a similar argument one can observe that the diffeomorphism σ_- sends $K_{(a,b)}$ to K_D . By homological reasons as we mentioned above the cases D and $-D$ occur only if $q = 1$ or $p - 1$ and since σ_- exists only when $q^2 \equiv -1$, $K_{(a,b)}$ is not isotopic to K_D or K_{-D} via a diffeomorphism which is isotopic to the identity. Also, we want to remark that there is no diffeomorphism of $L(p, q)$ sending $K_{(a,b)}$ to B or $-B$. If there was such a diffeomorphism, then by [4], it would be σ_+ or σ_- or τ . From above we see that it cannot be σ_+ or σ_- . Therefore, it must be τ but the fact that $\tau^2 = id$ gives us a contradiction.

Now, using Theorem 2.4 we observe that when $p = 2$ the diffeomorphisms τ and σ_+ are isotopic to the identity and hence we have Case (3). In Case 2, when $p \neq 2$ and $q = 1$ or $p - 1$ the knots $K_{(a,b)}$ and K_C are isotopic since in this case only σ_+ is isotopic to the identity. In the remaining cases, only τ exists and when $p \neq 2$, τ is not isotopic to the identity. This proves Case (1). \square

Lemma 2.5. *A torus knot $K_{(a,b)}$ in $L(p, q)$ has a rational Seifert surface $S_{K_{(a,b)}}$ of Euler characteristic*

$$\chi(S_{K_{(a,b)}}) = \frac{|rb| + (1 - |rb|)|rap + rbq|}{p}$$

where r is the order of $K_{(a,b)}$.

Proof. Let $K_{(a,b)}$ be a rationally null-homologous torus knot of order r in $L(p, q)$. We may construct a rational Seifert surface $S_{K_{(a,b)}}$ for r copies of $K_{(a,b)}$ as follows: First recall that $r[K] = m[\mu_1] + l[\mu_2] = m(-q[\mu_2] + p[\lambda_2]) + l[\mu_2]$ in $H_1(\partial V_2)$ where $m = \frac{rb}{p}$, $l = ra + \frac{rb}{p}q$ and $p \mid rb$. Construct the rational Seifert surface $S_{K_{(a,b)}}$ by taking $|m|$ parallel copies of the meridional disk μ_1 of ∂V_1 and $|l|$ parallel copies of the meridional disk μ_2 of ∂V_2 and then attaching a haft twisted band at each intersection for a total number of $p|l||m| = |l||rb|$ bands. Then, the Euler characteristic $\chi(S_{K_{(a,b)}})$ of $S_{K_{(a,b)}}$ is $\chi(S_{K_{(a,b)}}) = \#(\text{disks}) - \#(\text{bands})$:

$$\begin{aligned} \chi(S_{K_{(a,b)}}) &= |l| + |m| - |l||rb| \\ &= \left|ra + \frac{rbq}{p}\right| + \frac{|rb|}{p} - \left|ra + \frac{rbq}{p}\right||rb| \\ &= \frac{|rb| + (1 - |rb|)|rap + rbq|}{p}. \end{aligned}$$

\square

Also, see [2, Lemma 2.3] for rational Seifert surface construction for torus knots.

3. CONVEX SURFACES

A closed oriented surface Σ in a contact 3-manifold is called *convex* if there is a *contact vector field* v , that is a vector field whose flow preserves the contact structure ξ , transverse to Σ . Given a convex surface Σ in a contact 3-manifold with a contact vector field v , the *dividing set* Γ_Σ of Σ is defined as $\Gamma_\Sigma = \{x \in \Sigma : v(x) \in \xi_x\}$. The dividing set Γ_Σ is a multi-curve, possibly disconnected and possibly with boundary. The dividing set Γ_Σ is transverse to the characteristic foliation, $\Sigma \setminus \Gamma_\Sigma = \Sigma_+ \sqcup \Sigma_-$ and there is a vector field v that expands/contracts a volume form w on Σ_+/Σ_- and v points out of Σ_+ .

Theorem 3.1 (Giroux's tightness criterion). *A convex surface Σ in a contact 3-manifold has a tight neighborhood if and only if $\Sigma \neq S^2$ and Γ_Σ has no homotopically trivial dividing curves or $\Sigma = S^2$ and Γ_Σ is connected.*

For more information and details, see [12], [15].

3.1. Legendrian knots. *The positive/negative stabilization $S_+(L)/S_-(L)$ of a Legendrian knot L in the standard tight contact structure on \mathbb{R}^3 is obtained by modifying the front projection of L by adding a down cusp/an up cusp as in Figure 1, respectively. Since stabilizations are done locally, by Darboux's theorem this defines stabilizations of Legendrian knots in any contact 3-manifold.*

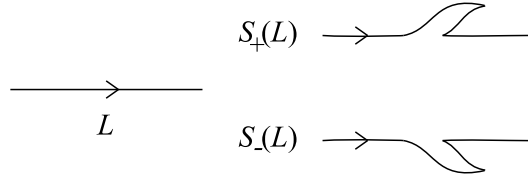


FIGURE 1. The positive stabilization $S_+(L)$ and the negative stabilization $S_-(L)$ of L .

The classical invariants of Legendrian knots are the topological knot type, the Thurston-Bennequin invariant $tb(L)$ and the rotation number $rot(L)$. The *Thurston-Bennequin invariant* measures the contact framing with respect to the Seifert framing and the *rotation number* of an oriented null-homologous Legendrian knot L can be computed as the winding number of TL after trivializing the contact structure along a Seifert surface for L . After stabilizing a Legendrian knot, the classical invariants change as $tb(S_{\pm}(L)) = tb(L) \pm 1$ and $rot(S_{\pm}(L)) = rot(L) \pm 1$.

Proposition 3.2 (Kanda [17, Proposition 4.5]). *Let L be a Legendrian curve on a surface Σ and let $tw_{\Sigma}(L)$ denote the twisting of the contact planes along L measured with respect to the framing on L given by Σ . Then Σ may be made convex relative to L if and only if $tw_{\Sigma}(L) \leq 0$. If Σ is a convex surface with dividing curves Γ , then*

$$(1) \quad tw_{\Sigma}(L) = -\frac{1}{2}\#(L \cap \Gamma).$$

Moreover, if Σ is a Seifert surface of L , the above formula computes the Thurston-Bennequin invariant $tb(L)$ of L and in this case the rotation number $rot(L)$ of L is

$$(2) \quad rot(L) = \chi(\Sigma_+) - \chi(\Sigma_-).$$

Also, from above proposition it follows that

Lemma 3.3. *A surface Σ with boundary may be made convex if and only if the twisting of contact planes along each boundary component is less than or equal to zero.*

3.2. Convex Torus in Standard Form. By Giroux's tightness criterion, a convex torus in a tight contact 3-manifolds has dividing set consists of $2n$ closed, parallel, homotopically non-trivial curves. A *convex torus in standard form* is a torus of slope s with characteristic foliation that consists of a 1-parameter family of curves of singularities with slope s , called *Legendrian divides*, and other 1-parameter family of curves with slope $r \neq s$, called *Legendrian rulings*. By Giroux's flexibility theorem, [12], [15], any convex torus with slope s in a tight contact 3-manifold can be put in a standard form with any ruling slope $r \neq s$.

Theorem 3.4 (Classification of tight contact structures on solid torus, [12], [15]). *There are $|(r_0+1) \cdots (r_{k-1}+1)(r_k)|$ tight contact structures on a solid torus $S^1 \times D^2$ with standard convex boundary having two dividing curves of slope $-\frac{p}{q}$, where $p > q > 0$ and $-\frac{p}{q} = r_0 - \frac{1}{r_1 - \frac{1}{r_2 - \cdots - \frac{1}{r_k}}}$. Moreover, all these contact structures are distinguished by the number of positive regions on a convex meridional disk with Legendrian boundary.*

Proposition 3.5 ([15, Proposition 4.16]). *Let ξ be a tight contact structure on $T^2 \times I$ with convex boundary having boundary slopes s_0 and s_1 on the boundary. Then for any s between s_0 and s_1 , there is a convex torus parallel to the boundary of $T^2 \times I$ with slope s .*

3.3. Bypasses. Let Σ be a convex surface in a contact 3-manifold, a bypass for Σ is a convex half disk D with Legendrian boundary such that

- (1) $\partial D = \gamma_0 \cup \gamma_1$, γ_0, γ_1 are two arcs that intersect at their end points,
- (2) $D \cap \Sigma = \gamma_0$,
- (3) the characteristic foliation of D has three elliptic singularities along γ_0 , two positive elliptic singularities at the end points of γ_0 and one negative elliptic singularity on the interior of γ_0 , and only positive singularities along γ_1 , alternating between positive elliptic and positive hyperbolic singularities,
- (4) γ_0 intersect Γ_Σ exactly at three elliptic singularities of γ_0 .

Figure 2 is a diagram illustrating a bypass disk.

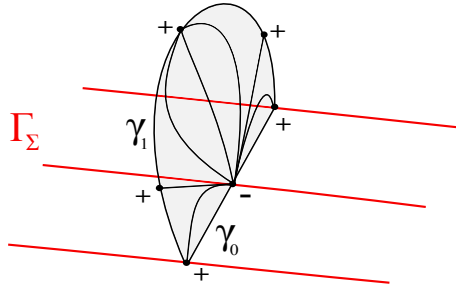


FIGURE 2. A bypass disk D .

A dividing curve $\gamma \subset \Gamma_\Sigma$ is called *boundary parallel* if γ cuts off a half disk which contains no other component of Γ_Σ in its interior. As the following propositions show, a boundary parallel curve allows us to find bypasses.

Proposition 3.6 (Imbalance Principle, [15, Proposition 3.17]). *Let $\Sigma = S^1 \times [0, 1]$ be a convex annulus with Legendrian boundary embedded in a tight contact 3-manifold. If $tw_\Sigma(S^1 \times \{0\}) < tw_\Sigma(S^1 \times \{1\}) \leq 0$, then there exists a boundary parallel curve and hence a bypass along $S^1 \times \{0\}$.*

Proposition 3.7 (Honda [15, Proposition 3.18]). *Let Σ be a convex surface with Legendrian boundary. If the dividing set Γ_Σ contains a boundary parallel dividing curve γ , then there exists a bypass for Σ , provided that Σ is not a disk with $tb(\partial\Sigma) = -1$.*

Remark 3.8. Note that an orientable Legendrian knot L and its (positive/negative) stabilization L' cobound a bypass disk. The stabilization L' of L is obtained by pushing L across this bypass disk. Therefore, locating bypasses, in particular boundary parallel curves are helpful in showing that a Legendrian knot destabilizes.

3.4. Relative Euler classes. We will use the relative Euler class of a $T^2 \times [0, 1]$ region in our calculations. Let ξ be a tight contact structure on $M = T^2 \times [0, 1]$ with convex boundary in standard form. Assume $\xi|_{\partial M}$ is trivializable and let s be a nowhere vanishing section of $\xi|_{\partial M}$. The relative Euler class $e(\xi, s) \in H_1(M, \partial M, \mathbb{Z})$ is the obstruction to extend s to all of M . If γ is an oriented Legendrian curve on $T^2 \times \{0\}$ and assume the annulus $A = \gamma \times [0, 1]$ is convex and has Legendrian boundary, then $e(\gamma) \equiv e(A) = \chi(A_+) - \chi(A_-)$ where A_\pm are the positive and negative regions of A determined by the dividing set of A .

4. LEGENDRIAN TORUS KNOTS IN LENS SPACES

A contact structure on a 3-manifold is *universally tight* if its pullback to the universal cover is tight. In this section, we classify Legendrian torus knots $L_{(a,b)}$ of knot type $K_{(a,b)}$ in universally tight contact structures on a lens space $L(p, q)$, where $p > q > 0$. We identify $L(p, q)$ as the quotient of $T^2 \times [0, 1]$ obtained by collapsing $y = \text{constant}$ curves on $T^2 \times \{0\}$ and collapsing $(-q, p)$ -curve on $T^2 \times \{1\}$ to a point. The universally tight contact structure ξ_{ut} on $L(p, q)$ is induced from minimally twisting universally tight contact structure on $T^2 \times [0, 1]$ with minimal number of dividing curves of slopes $s_0 = 0$ on $T^2 \times \{0\}$ and $s_1 = -\frac{p}{q}$ on $T^2 \times \{1\}$. There are two such universally tight contact structures on $T^2 \times [0, 1]$ and they satisfy $PD(e(\xi, s)) = \pm((-q, p) - (-1, 1))$, [15, Proposition 5.1]. We assume that ξ_{ut} is induced from the universally tight contact structure with $PD(e(\xi, s)) = (-q, p) - (-1, 1)$. The results in this section similarly hold for the other case and can be easily written down. There are exactly two universally tight contact structure on $L(p, q)$ when $q \neq p - 1$ and there is only one when $q = p - 1$, [15, Proposition 5.1].

Remark 4.1. Note that, by Proposition 3.5 in a universally tight Lens space $L(p, q)$, we can find a convex torus T with dividing curves of any slope in $(-\frac{p}{q}, 0)$.

The definition of Thurston-Bennequin invariant and the rotation number can be extended for rationally null-homologous Legendrian knots. They are defined and studied in [1]. Also rational Thurston-Bennequin invariant has been studied for knots in Lens spaces in [3] and for links in rational homology spheres in [18].

The *rational Thurston-Bennequin invariant* measures the contact framing of a rationally null-homologous Legendrian knot with respect to the rational Seifert framing of the knot. Let $L_{(a,b)}$ be a Legendrian torus knot of knot type $K_{(a,b)}$ of order r in the universally tight contact structure ξ_{ut} on $L(p, q)$. Now recall the rational Seifert surface $S_{K_{(a,b)}}$ construction for r copies of $K_{(a,b)}$ in the proof of Lemma 2.5. The rational Seifert framing of $L_{(a,b)}$ coming from $S_{K_{(a,b)}}$ is $\frac{1}{r} \frac{plm}{r} = \frac{1}{r} lb$ where $m = \frac{rb}{p}$, $l = ra + \frac{rb}{p}q$. Furthermore, by Equation (1) in Proposition 3.2, by using the set of dividing curves Γ for the Heegaard torus containing $L_{(a,b)}$, the contact framing of $L_{(a,b)}$ can be computed as $-\frac{1}{2} \#(L_{(a,b)} \cap \Gamma)$. Here $\#(L_{(a,b)} \cap \Gamma)$ is the unsigned count of intersection number of $L_{(a,b)}$ and Γ . Note that arbitrary (a, b) -curve and (c, d) -curve on a torus intersect $|\det \begin{pmatrix} a & c \\ b & d \end{pmatrix}|$ times. If the dividing curves Γ have slope $-\frac{t}{s}$ and if $2n$ is the number of dividing curves, then the rational Thurston-Bennequin invariant of a Legendrian torus knot $L_{(a,b)}$ is

$$tb_{\mathbb{Q}}(L_{(a,b)}) = \frac{1}{r} lb - n |\det \begin{pmatrix} a & -s \\ b & t \end{pmatrix}|.$$

Let $\mathcal{L}(K)$ denote the set of all rationally null-homologous Legendrian knots in knot type K . The *maximal rational Thurston-Bennequin invariant* $\overline{tb}_{\mathbb{Q}}(K)$ of the knot type K is defined as

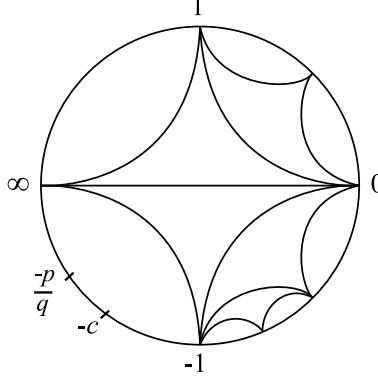
$$\overline{tb}_{\mathbb{Q}}(K) = \max\{tb_{\mathbb{Q}}(L) \mid L \in \mathcal{L}(K)\}.$$

Theorem 4.2. For a, b relatively prime integers, the maximal rational Thurston-Bennequin invariant $\overline{tb}_{\mathbb{Q}}(K_{(a,b)})$ is

- (1) $\overline{tb}_{\mathbb{Q}}(K_{(a,b)}) = \frac{1}{r} lb - (a + b)$ if $a, b \geq 0$,
- (2) $\overline{tb}_{\mathbb{Q}}(K_{(a,b)}) = \frac{1}{r} lb - |ac + b|$ if $-\infty < \frac{b}{a} \leq -\frac{p}{q}$,
- (3) $\overline{tb}_{\mathbb{Q}}(K_{(a,b)}) = \frac{1}{r} lb$ if $-\frac{p}{q} < \frac{b}{a} < 0$,

where $l = ra + \frac{rb}{p}q$, $-c = \lfloor -\frac{p}{q} \rfloor + 1$ and r is the order of $K_{(a,b)}$.

Proof. (1) By Remark 4.1, we know that we can find a convex torus T with dividing curves of any slope in $(-\frac{p}{q}, 0)$. In particular, there is a convex torus T with two dividing curves of slope -1 which contains a Legendrian knot $L_{(a,b)}$ of knot type $K_{(a,b)}$ as a Legendrian ruling curve.

FIGURE 3. $-c = \lfloor -\frac{p}{q} \rfloor + 1$

For $a, b \geq 0$, $\#(L_{(a,b)} \cap \Gamma)$ is minimal on T . Therefore, $\overline{tb}_{\mathbb{Q}}(K_{(a,b)}) = \frac{1}{r}lb - |\det \begin{pmatrix} a & -1 \\ b & 1 \end{pmatrix}| = \frac{1}{r}lb - (a + b)$.

(2) In the case when $-\infty < \frac{b}{a} \leq -\frac{p}{q}$, $\#(L_{(a,b)} \cap \Gamma)$ is minimal on a convex torus T with two dividing curves of slope $-c = \lfloor -\frac{p}{q} \rfloor + 1$. Then, $\overline{tb}_{\mathbb{Q}}(K_{(a,b)}) = \frac{1}{r}lb - |\det \begin{pmatrix} a & -1 \\ b & c \end{pmatrix}| = \frac{1}{r}lb - |ac + b|$.

(3) When $-\frac{p}{q} < \frac{b}{a} < 0$, by Remark 4.1 again there is a convex torus T with two dividing curves of slope $\frac{b}{a}$ and T contains $L_{(a,b)}$ as a Legendrian divide. This implies that the contact framing of $L_{(a,b)}$ is 0 and thus $\overline{tb}_{\mathbb{Q}}(K_{(a,b)}) = \frac{1}{r}lb$. \square

The *rational rotation number* $rot(L)$ of an oriented rationally null-homologous Legendrian knot L of order r can be computed as the winding number of TL after trivializing the contact structure along a rational Seifert surface for L divided by r .

Let $L_{(a,b)}$ be a Legendrian torus knot of order r with maximal rational Thurston-Bennequin invariant that sits on a Heegaard torus T in $(L(p, q), \xi_{ut})$. In what follows, we will explain how to compute the rational rotation number $rot_{\mathbb{Q}}(L_{(a,b)})$ of $L_{(a,b)}$ in a similar way as Etnyre and Honda computed for Legendrian torus knots in standard tight S^3 in [8]. Let $L(p, q) = V_1 \cup_T V_2$ where V_1 and V_2 are both $D^2 \times S^1$ with meridional curve μ_1 and μ_2 respectively. Define an invariant of homology classes of curves on Heegaard torus T as follows: Let v be any globally non-zero section of ξ_{ut} and w a section of $\xi_{ut}|_T$ which is tangent to the Legendrian divides and transverse to and twists along the Legendrian ruling curves. Let $f_T(\gamma)$ equal to the rotation of v relative w along a closed oriented curve γ on T . If L is a ruling curve or a Legendrian divide on T then $f_T(L) = r \, rot(L)$. For details and the properties of the function f_T , see [7] and [8].

The rational rotation number of a Legendrian torus knot $L_{(a,b)}$ on the Heegaard torus $\partial V_2 = T$ can be computed as

$$(3) \quad r \operatorname{rot}_{\mathbb{Q}}(L_{(a,b)}) = m f_T(\mu_1) + l f_T(\mu_2)$$

where $m = \frac{rb}{p}$, $l = ra + \frac{rb}{p}q$ and r is the order of $L_{(a,b)}$.

Theorem 4.3. *Let $L_{(a,b)}$ be a Legendrian torus knot with maximal rational Thurston-Bennequin invariant. The range of possible rational rotation numbers $\operatorname{rot}_{\mathbb{Q}}(L_{(a,b)})$ of $L_{(a,b)}$ is*

- (1) $\{\frac{1}{r}m((p-q)-1)\}$ if $a, b \geq 0$,
- (2) $\{\frac{1}{r}(m((p-cq)-1) + l(c-1))\}$ if $-\infty < \frac{b}{a} \leq -\frac{p}{q}$,
- (3) if $-\frac{p}{q} < \frac{b}{a} < 0$, we have
 - $\{\frac{1}{r}(\pm(m|pa+qb| \pm(m-2mk)) + l(1-|b|))\}$, $k \in \mathbb{Z}$, $0 \leq k \leq pn-q$, $|a| = |b|n + e$ if $-1 \leq \frac{b}{a} < 0$,
 - $\{\frac{1}{r}(m(|pa+qb|-1) + l(|b|-1))\}$ if $-\frac{p''}{q''} < \frac{b}{a} < -1$,
 - $\{\frac{1}{r}(\pm(m|pa+qb| \pm(m-2mk)) + l(1-|b|))\}$, $k \in \mathbb{Z}$, $0 \leq k \leq n$, $|pa+qb| = |p''a + q''b|n + e$ if $-\frac{p}{q} < \frac{b}{a} \leq -\frac{p''}{q''}$,

where $m = \frac{rb}{p}$, $l = ra + \frac{rb}{p}q$, $-c = \lfloor -\frac{p}{q} \rfloor + 1$, r is the order of $L_{(a,b)}$ and $-\frac{p''}{q''}$ is the point on $\partial \mathbb{H}^2$ which is closest to -1 and has an edge to $-\frac{p}{q}$.

Proof. (1) If $a, b \geq 0$, we know that Legendrian torus knot $L_{(a,b)}$ with maximal rational Thurston-Bennequin invariant is on a standard convex torus T of slope -1 in $L(p, q) = V_1 \cup_T V_2$. To compute the rational rotation number for $L_{(a,b)}$, we need to compute $f_T(\mu_1)$ and $f_T(\mu_2)$.

For $f_T(\mu_1)$, consider the meridional disk D_{V_1} of V_1 . We may isotope D_{V_1} to be convex relative to μ_1 . We can do this by arranging the Legendrian ruling curves on T to be $(-q, p)$ -curves. Then we see that the twisting of the contact planes along μ_1 will be less than or equal to zero; and hence by Lemma 3.3, we can make D_{V_1} convex. Now since the dividing curves on T are $(-1, 1)$ -curves and intersect μ_1 , $2(p-q)$ times, the dividing curves on D_{V_1} intersect μ_1 , $2(p-q)$ times. The key observation here is that the dividing curves on the meridional disk D_{V_1} separate off disks of the same sign, positive sign, that contain no other dividing curves. A way to see that is by considering a solid torus V containing V_1 and by looking at the annulus $A = \overline{D} \setminus D_{V_1}$ where D is the meridional disk for V . If the dividing curves were not as claimed, then there would be bypasses of both signs on D_{V_1} . Then, we would glue one of the bypasses to a bypass of the same sign on A and that would result in an overtwisted disk. Therefore, the dividing curves are as claimed. To match the bypasses on D_{V_1} and A one may use Slide Maneuvers trick, see [15]. Lastly, note that on D_{V_1} all the disks, bypasses, have positive sign since we fix the universally tight contact structure on $L(p, q)$ in this way. Then, by Equation (2) in Proposition 3.2, we have $f_T(\mu_1) = (p-q)-1$.

For $f_T(\mu_2)$, we will argue in the same way as above by considering the meridional disk D_{V_2} of V_2 . The dividing curves on D_{V_2} intersect μ_2 , 2 times since the dividing curves on T intersect μ_2 , 2 times. Thus, we have only one possible configuration for the dividing curves and by Equation (2) in Proposition 3.2, $f_T(\mu_2) = 0$. This proves (1).

(2) If $-1 < \frac{b}{a} < 0$, this time $L_{(a,b)}$ with maximal rational Thurston-Bennequin invariant is on a standard convex torus T of slope $-c$ where $-c = \lfloor -\frac{p}{q} \rfloor + 1$. Thus, the dividing curves on T intersect μ_2 , $2c$ times and intersect μ_1 , $2(p - cq)$ times. By the same reasons as in Case (1), $f_T(\mu_2) = c - 1$ and $f_T(\mu_1) = (p - cq) - 1$. Then by Equation (3), $rrot_{\mathbb{Q}}(L_{(a,b)}) = mf_T(\mu_1) + lf_T(\mu_2) = m((p - cq) - 1) + l(c - 1)$.

(3) When $-\frac{p}{q} < \frac{b}{a} < 0$, by Remark 4.1 there is a convex torus T with two dividing curves of slope $\frac{b}{a}$ which contains $L_{(a,b)}$ as a Legendrian divide in $L(p, q) = V_1 \cup_T V_2$. We have three subcases in this last case. In Case (i), when $-1 \leq \frac{b}{a} < 0$, and in Case (iii), when $-\frac{p}{q} < \frac{b}{a} \leq -\frac{p''}{q''}$, the convex torus T falls in a solid torus region. In Case (ii), when $-\frac{p''}{q''} < \frac{b}{a} < -1$, the convex torus T falls in a positive $T^2 \times I$ region. By positive $T^2 \times I$ region we mean that if $T^2 \times I$ decomposed into basic slices, (see [15] for a description), then all have the same sign, positive sign. All basic slices have the same sign since we work in a universally tight contact structure and they all have the positive sign since from the beginning we fix the universally tight contact structure in this way.

Case (i), when $-1 < \frac{b}{a} < 0$, by Proposition 3.5, inside V_1 there is a solid torus V_{n+1} with two dividing curves of slope $-\frac{1}{n+1}$ and there is a solid torus V_n containing V_1 with two dividing slope of $-\frac{1}{n}$, where $|a| = n|b| + e$. Set $T_{n+1} = \partial V_{n+1}$ and $T_n = \partial V_n$. In this case, the solid torus V_2 is contained in V_{n+1} and V_2 contains V_n . To compute possible rational rotation numbers we need to compute the followings: $f_T(\mu_1)$, $f_T(\mu_2)$, and in this case to compute $f_T(\mu_1)$ we need the possible values for $f_{T_n}(\mu'_1)$ where μ'_1 is the boundary of the meridional disk of $\overline{L(p, q) \setminus V_n}$. Computations in this case and in Case (iii) are very similar to the computation of rotation numbers of negative Legendrian torus knots in [8]. Let us first compute $f_T(\mu_2)$.

$f_T(\mu_2) = 1 - |b|$ or $|b| - 1$: Let $D_{V_{n+1}}$ and D_{V_2} be convex meridional disks for V_{n+1} and V_2 respectively (if necessary isotope the disks to be convex by arranging Legendrian rulings to be meridional and by using Lemma 3.3) and let $D_{V_{n+1}} = A \cup D_{V_2}$ where $A = \overline{D_{V_{n+1}} \setminus D_{V_2}}$. The dividing curves $\Gamma_{T_{n+1}}$ on $T_{n+1} = \partial V_{n+1}$ are $(-(n+1), 1)$ -curves and intersect $\partial D_{V_{n+1}}$, 2 times. Moreover, the dividing curves Γ_T on $T = \partial V_2$ are (a, b) -curves and hence intersect $\mu_2 = \partial D_{V_2}$, $2|b|$ times. Note that by the same reasons as in Case (1) the dividing curves on D_{V_2} separate off disks of the same sign and contain no other dividing curves. Then, by Equation (2) in Proposition 3.2, $f_T(\mu_2) = 1 - |b|$ or $|b| - 1$.

$f_{T_n}(\mu'_1) \in \{pn - q - 1, pn - q - 3, \dots, 3 + q - pn, 1 + q - pn\}$: Let D' be a convex meridional disk for $\overline{L(p, q)} \setminus V_n$ and let $\mu'_1 = \partial D'$. Note that μ'_1 is a $(-q, p)$ -curve on T . Therefore, the dividing curves Γ_{T_n} on T_n intersect the boundary of D' , $2(pn - q)$ times and between each two adjacent points in $\Gamma_{T_n} \cap \partial D'$ there is one point in $\Gamma_{D'} \cap \partial D'$ where $\Gamma_{D'}$ denotes the dividing curves on D' . Also, by Giroux's tightness criterion, there are no homotopically trivial dividing curves on D' . It follows that, there are exactly $(pn - q)$ dividing curves on D' . By Equation (2) in Proposition 3.2 and by examining possible configurations of dividing curves of D' , we have $f_{T_n}(\mu'_2) \in \{(pn - q) - 1, (pn - q) - 3, \dots, 3 - (pn - q), 1 - (pn - q)\}$.

If $f_T(\mu_2) = 1 - |b|$, then $f_T(\mu_1) = f_{T_n}(\mu'_1) + pn - q - |ap + qb|$: Let $e_{T^2 \times I}$ be the relative Euler class for the region $T^2 \times I$ between T and T_n . Then, $e_{T^2 \times I}(\mu_1) = f_{T_n}(\mu'_1) - f_T(\mu_1)$. We know the possible values of $f_{T_n}(\mu'_1)$ from (b), now to compute $f_T(\mu_1)$ we need to compute $e_{T^2 \times I}(\mu_1)$.

First, recall that μ_1 is a $(-q, p)$ -curve on T . Start with a convex annulus A of slope $-\frac{p}{q}$ between T_{n+1} and T_n with Legendrian boundary. Make the Legendrian ruling curves on T to be of slope $-\frac{p}{q}$ and consider subannulus A_n , between T and T_n , and subannulus A_{n+1} , between T_{n+1} and T . Note that the dividing curves Γ_A on A intersect $T_n \cap A$ in $2(pn - q)$ points, T in $2|pa + qb|$ points and $T_{n+1} \cap A$ in $2(pn + p - q)$ points. Note also that $2(pn - q)$ dividing curves run from one boundary component to the other boundary component of A . Therefore, on A_n , $(pn - q)$ dividing curves run from one boundary component to the other boundary component and there are $|pa + qb| - pn - q$ other dividing curves whose boundaries are on T . See Fact 3 in [8] for details. These $|ap + qb| - pn - q$ dividing curves on A_n separate off disks that contain no other dividing curves and the disks are of the same sign, otherwise by arguing in the same way as in Case(1), we can find an overtwisted disk in $L(p, q)$. Thus, $e_{T^2 \times I}(\mu_1) = |pa + qb| - (pn - q)$ or $(pn - q) - |pa + qb|$.

Now, we will show that when $f_T(\mu_2) = 1 - |b|$, $e_{T^2 \times I}(\mu_1) = pn - q - |pa + qb|$. Make the Legendrian curves on T_n to be slope $\frac{b}{a}$ curves and consider the convex annulus A' of slope $\frac{b}{a}$ between T and T_n . The dividing curves $\Gamma_{A'}$ on A' intersect only $T_n \cap A'$ in $2|a + bn| = 2e$ points. Thus, the Euler class of the region equals e or $-e$ (since there are no nested bypasses). On the other hand, the Euler class of this region can be calculated as $e_{T^2 \times I}(m\mu_1 + l\mu_2) = \frac{1}{r}(me_{T^2 \times I}(\mu_1) + le_{T^2 \times I}(\mu_2))$. Note that $e_{T^2 \times I}(\mu_2) = f_{T_n}(\mu_2) - f_T(\mu_2) = -f_T(\mu_2) = 1 - |b|$ since $f_{T_n}(\mu_2) = 0$. If we assume that $e_{T^2 \times I}(\mu_1) = |pa + qb| - (pn - q)$, then $|e_{T^2 \times I}(m\mu_1 + l\mu_2)| \neq e$. Therefore, $e_{T^2 \times I}(\mu_1) = pn + q - |ap + qb|$. Note that if $e_{T^2 \times I}(\mu_1) = pn - q - |pa + qb|$, then in when $a > 0$, $b < 0$, we have $e_{T^2 \times I}(m\mu_1 + l\mu_2) = -e$ and when $a < 0$, $b > 0$, we have $e_{T^2 \times I}(m\mu_1 + l\mu_2) = e$.

Now by Equation (3), we have

$$\begin{aligned} r \operatorname{rot}_{\mathbb{Q}}(L_{(a,b)}) &= m f_T(\mu_1) + l f_T(\mu_2) \\ &= m(f_{T_n}(\mu'_1) + pn - q - |pa + qb|) + l(1 - |b|) \\ &= -m|pa + qb| - (m - 2mk) + l(1 - |b|), \end{aligned}$$

where $k \in \mathbb{Z}, 0 \leq k \leq pn - q$.

If $f_T(\mu_1) = |b| - 1$, then $f_T(\mu_2) = f_{T_n}(\mu'_1) + |pa + qb| - (pn - q)$: This case is similar to the previous case, for this reason we left the details to the reader. In this case, by Equation (3), we have

$$\begin{aligned} r \operatorname{rot}_{\mathbb{Q}}(L_{(a,b)}) &= m f_T(\mu_1) + l f_T(\mu_2) \\ &= m(f_{T_n}(\mu'_1) + |pa + qb| - (pn - q)) + l(|b| - 1) \\ &= m|pa + qb| + (m - 2mk) + l(|b| - 1), \end{aligned}$$

where $k \in \mathbb{Z}, 0 \leq k \leq pn - q$.

This proves Case (i) of Case (3).

Case (ii), when $-\frac{p''}{q''} < \frac{b}{a} < -1$, let T be a standard convex torus of slope $\frac{b}{a}$ on which $L_{(a,b)}$ sits as a Legendrian divide. Unlike the previous case, we have only possible rational rotation number in this case. The dividing curves intersect μ_1 , $2|pa + qb|$ times and intersect μ_2 , $2|b|$ times. By the same arguments as in Case (1), we have $f_T(\mu_1) = |pa + qb| - 1$ and $f_T(\mu_2) = |b| - 1$. Then by Equation 3, we have $r \operatorname{rot}_{\mathbb{Q}}(L_{(a,b)}) = m f_T(\mu_1) + l f_T(\mu_2) = m(|pa + qb| - 1) + l(|b| - 1)$.

For the last case of Case (3), when $-\frac{p}{q} < \frac{b}{a} < -\frac{p''}{q''}$, let T be a standard convex torus with slope $\frac{b}{a}$ on which $L_{(a,b)}$ sits as a Legendrian divide and let $L(p, q) = V_1 \cup_T V_2$. The important observation here is that inside V_1 there is a solid torus V_{n+1} with two dividing curves of slope $-\frac{p+(n+1)p''}{q+(n+1)q''}$ and there is a solid torus V_n containing V_1 with two dividing slope of $-\frac{p+np''}{q+nq''}$, where $|pa + qb| = |p''a + q''b|n + e$.

Now, set $T_{n+1} = \partial V_{n+1}$ and $T_n = \partial V_n$. To compute possible rational rotation numbers we need to compute the followings: $f_T(\mu_1)$, $f_T(\mu_2)$, and to compute $f_T(\mu_1)$ we need the possible values for $f_{T_n}(\mu'_1)$ where μ'_1 is the boundary of the meridional disk of $\overline{L(p, q)} \setminus V_n$. These computations are very similar to the previous cases and left to the reader.

(a) $f_T(\mu_2) = 1 - |b|$ or $|b| - 1$,

(b) $f_{T_n}(\mu'_1) \in \{n - 1, n - 3, \dots, 1 - n\}$. Here note that the dividing curves Γ_{T_n} of T_n are $(-(q + nq''), p + np'')$ -curve and intersect μ'_1 (which is a $(-q, p)$ -curve), $2|\det \begin{pmatrix} -(q + nq'') & p + np'' \\ -q & p \end{pmatrix}| = 2|n(qp'' - pq'')| = 2n$ times. Note

that $qp'' - pq'' = 1$ since $-\frac{p}{q}$ has an edge to $-\frac{p''}{q''}$.

(c) If $f_T(\mu_2) = 1 - |b|$, then $f_T(\mu_1) = f_{T_n}(\mu'_1) + n - |pa + qb|$,

(d) If $f_T(\mu_2) = |b| - 1$, then $f_T(\mu_1) = f_{T_n}(\mu'_1) + |pa + qb| - n$.

□

Theorem 4.4. *Legendrian torus knots are uniquely realized up to contactomorphism if and only if they have the same knot type, the same rational Thurston-Bennequin invariant and the same rational rotation number.*

The proof of the Theorem 4.4 follows from the following lemmas.

Lemma 4.5. *Two Legendrian (a, b) -torus knots L and L' in $(L(p, q), \xi_{ut})$ with same maximal rational Thurston-Bennequin invariant are uniquely realized up to contactomorphism if and only if $\text{rot}_{\mathbb{Q}}(L) = \text{rot}_{\mathbb{Q}}(L')$.*

Proof. Let T and T' be standard convex tori on which L and L' respectively sit in $L(p, q)$. Also, let $V_1 \cup_T V_2$ and $V'_1 \cup_{T'} V'_2$ be the Heegaard splittings associated to T and T' . Since $\overline{tb}_{\mathbb{Q}}(L) = \overline{tb}_{\mathbb{Q}}(L')$, the slopes of the dividing curves on T and T' are the same. Then, by Theorem 3.4, by the classification of tight contact structures on solid tori, there is a contactomorphism $\phi : V_1 \rightarrow V'_1$ such that $\phi(L) = L'$. By Theorem 3.4 again, the contactomorphism type of a tight contact structure on V_2 or V'_2 is determined by the number of positive bypasses on meridional disks. If r is the order of L and L' in $L(p, q)$, then the number of positive bypasses on meridional disks are determined by r times the rational rotation number of the Legendrian knots L and L' , respectively. Therefore, we can extend the contactomorphism ϕ to all of $L(p, q)$ provided that L and L' have the same rational rotation number. \square

Lemma 4.6. *If $L_{(a,b)}$ is a Legendrian torus knot in $(L(p, q), \xi_{ut})$ with non-maximal rational Thurston-Bennequin invariant then there is a Legendrian torus knot $L'_{(a,b)}$ such that $L_{(a,b)}$ is a stabilization of $L'_{(a,b)}$.*

Proof. For the proof we have three cases, we will explain Case(1) in detail, other cases are quite similar to this case.

Case (1), $a, b \geq 0$. Let T be a standard convex torus on which the Legendrian torus knot $L_{(a,b)}$ sits. Since $tb_{\mathbb{Q}}(L) < \overline{tb}_{\mathbb{Q}}(L_{(a,b)})$, the dividing curves Γ_T on T have slope $-\frac{t}{s} \neq -1$. Recall that when $a, b \geq 0$, $L_{(a,b)}$ with maximal rational Thurston-Bennequin invariant lies on a convex torus with two dividing curves of slope -1 . By Remark 4.1, we know that we can find a convex torus T' with dividing curves of any slope in $(-\frac{p}{q}, 0)$. In particular, there is a convex torus T' with two dividing curves of slope -1 . Now take the $T \times [0, 1]$ region between T and T' and take the annulus $A = L_{(a,b)} \times [0, 1]$. Furthermore, arrange the ruling curves on both boundary components of $T^2 \times I$ to be slope $\frac{b}{a}$. Then ∂A will be Legendrian ruling curves on the boundary of $T^2 \times I$ and the twisting of contact planes along each boundary component will be less than zero. Therefore by Lemma 3.3, we can make A convex. The dividing curves on $T = T^2 \times \{0\}$ intersect A in $2n(at + sb)$ points, where n is the number of dividing curves and the dividing curves on $T' = T^2 \times \{1\}$ intersect A in $2(a + b)$ points. Since $2n(at + bs) > 2(a + b)$, there is a boundary parallel dividing curve along $T = T^2 \times \{0\}$ and hence by Proposition 3.7 a bypass for $L_{(a,b)}$. In other words, $L_{(a,b)}$ destabilizes.

Case(2), if $-\infty < \frac{b}{a} \leq -\frac{p}{q}$. In this case $L_{(a,b)}$ with maximal rational Thurston-Bennequin invariant lies on a convex torus T' with dividing slope $-c = \lfloor -\frac{p}{q} \rfloor + 1$ on which the intersection number of L and the dividing curves on T is minimal. Let $L_{(a,b)}$ be on a standard convex torus T with dividing slope $-\frac{t}{s} \neq -c$. As in the proof of Case(1) take a region $T^2 \times [0, 1]$ between T and T' and argue as in Case(1).

Case(3), if $-\frac{p}{q} < \frac{b}{a} < 0$. We know that $L_{(a,b)}$ with maximal rational Thurston-Bennequin invariant lies on a convex torus T' with dividing slope $\frac{b}{a}$. Let L lie on the convex torus T with dividing slope $-\frac{t}{s} \neq \frac{b}{a}$. Again, we take a region $T^2 \times [0, 1]$ between T and T' and argue as in Case(1). \square

Note that the knot type $K_{(a,b)}$ has a unique Legendrian realization with maximal ratioanal Thurston-Bennequin invariant in the following cases when $a, b \geq 0$ or $-\infty < \frac{b}{a} \leq -\frac{p}{q}$ or when $-\frac{p''}{q''} < \frac{b}{a} < -1$. Therefore in these cases, a Legendrian torus knot with non-maximal rational Thurston-Bennequin invariant destabilizes to the unique Legendrian torus knot with maximal rational Thurston-Bennequin invariant. For the remaining cases, when $-1 \leq \frac{b}{a} < 0$ or when $-\frac{p}{q} < \frac{b}{a} \leq -\frac{p''}{q''}$, take two Legendrian torus knots L and L' realizing $K_{(a,b)}$ with maximal rational Thurston-Bennequin invariant. Now let us understand the relationship between their stabilizations.

Lemma 4.7. *Let L and L' be (a, b) -torus knot with maximal rational Thurston-Bennequin invariant in $(L(p, q), \xi_{ut})$ where $-1 \leq \frac{b}{a} < 0$ or $-\frac{p}{q} < \frac{b}{a} \leq -\frac{p''}{q''}$. If the rational rotation numbers of L and L' are r and $r - 2e$ respectively, then $S_-^e(L)$ and $S_+^e(L')$ are Legendrian contactomorphic. If the rational rotation numbers of L and L' are r and $r - 2(q - e)$ respectively, then $S_-^{q-e}(L)$ and $S_+^{q-e}(L')$ are Legendrian contactomorphic. Here when $-1 \leq \frac{b}{a} < 0$, e is an integer such that $|a| = |b|n + e$ and when $-\frac{p}{q} < \frac{b}{a} \leq -\frac{p''}{q''}$, e satisfies $|pa + qb| = n|p''a + q''b| + e$.*

By two knots being Legendrian contactomorphic, we mean that there is a contactomorphism of the 3-manifold sending one knot to the other.

Proof. The proof is similar to the case for negative Legendrian knots in standard tight S^3 . The proof follows from examining the annulus A of slope $\frac{b}{a}$ between T and T_n . In the first case when $-1 \leq \frac{b}{a} < 0$, the dividing curves on T_n are of slope $-\frac{1}{n}$ and in the latter case when $-\frac{p}{q} < \frac{b}{a} \leq -\frac{p''}{q''}$, dividing curves on T_n are of slope $-\frac{p+np''}{q+nq''}$. The proof follows the proof of Lemma 4.12 in [8]. \square

5. FINAL REMARKS AND QUESTIONS

Remark 5.1. Legendrian knots can be classified up to contact isotopy or up to global contactomorphism. By Eliashberg, [6, Theorem 2.4.2], the group

of co-orientation preserving contactomorphisms of the standard tight S^3 is connected. Therefore, for Legendrian knots in standard tight S^3 , these two classifications are equivalent. However, for arbitrary tight contact closed 3-manifolds the group of co-orientation preserving contactomorphisms is not well understood. In particular, nothing is known for tight contact Lens spaces. It would be very interesting to know:

Question 1. *What can one say about the group $\pi_0(\text{Diff}(L(p, q), \xi))$ of path components of $\text{Diff}(L(p, q), \xi)$, where $\text{Diff}(L(p, q), \xi)$ denotes the group of contactomorphisms of $L(p, q)$ and ξ is a tight contact structure on $L(p, q)$?*

In particular,

Question 2. *Is the group of co-orientation preserving contactomorphisms of universally tight contact structures on Lens spaces connected?*

We want to remark that positive answer to Question 2 and Theorem 4.4 that we proved in previous section together provide us the classification of Legendrian torus knots up to Legendrian isotopy in universally tight Lens spaces.

Remark 5.2. Recall that a transverse knot in a contact 3-manifold is a knot which is everywhere transverse to the contact planes. There are two types of classical invariants for null-homologous transverse knots; knot type and self-linking number. One can define the rational self-linking number for a rationally null-homologous transverse knot using a rational Seifert surface of the knot, see [1]. From [8, Theorem 2.10], we know that two transverse knot in a contact 3-manifold are transversely isotopic if and only if their Legendrian push offs are Legendrian isotopic after each has been negatively stabilized some number of times. Note that, this is also true when we replace transverse isotopic and Legendrian isotopic by contactomorphic. As a consequence, from Theorem 4.4 we have

Theorem 5.3. *Transverse torus knots in universally tight contact structures on Lens Spaces are determined up to contactomorphism by their knot type and rational self-linking number.*

Another interesting question is

Question 3. *For what knot types in Lens spaces are all Legendrian realizations determined by their classical invariants?*

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